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Landau quantum systems: an approach based on symmetry

J Negro, M A del Olmo and A Rodríguez-Marco

Departamento de Física Teórica, Universidad de Valladolid, E-47011, Valladolid, Spain

E-mail: jnegro@fta.uva.es and olmo@fta.uva.es

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Abstract

We show that the Landau quantum systems (or integer quantum Hall effect systems) in a plane, sphere or a hyperboloid, can be explained in a complete and meaningful way by group-theoretical considerations concerning the symmetry group of the corresponding configuration space. The crucial point in our development is the role played by locality and its appropriate mathematical framework, the fibre bundles. In this way the Landau levels can be understood as the local equivalence classes of the symmetry group. We develop a unified treatment that supplies the correct geometric way to recover the planar case as a limit of the spherical or the hyperbolic quantum systems when the curvature goes to zero. This is an interesting case where a contraction procedure gives rise to nontrivial cohomology starting from a trivial one. We show how to reduce the quantum hyperbolic Landau problem to a Morse system using horocyclic coordinates. An algebraic analysis of the eigenvalue equation allows us to build ladder operators which can help in solving the spectrum under different boundary conditions.

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1. Introduction

The planar Landau levels arise in the frame of quantum mechanics (QM) when a charged particle evolves under the influence of an external constant magnetic field perpendicular to the plane [1]. Landau quantum systems can also be generalized to other surfaces with a normal stationary magnetic field. In this way, the spherical and hyperbolic Landau systems have been also studied [2], but there is still a lack of a comprehensive characterization of these systems from the point of view of their symmetry. We will try to fill the gap here by a systematic study of such kinds of quantum systems based on their spatial symmetries.

Widespread theoretical, as well as experimental activity, has been paid to two-dimensional (2D) quantum systems of charged particles in the last two decades. In particular,

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the quantum Hall effect [3], 2D systems of electrons subjected to strong external magnetic fields at very low temperatures, has received a lot of attention due to its interesting and surprising properties [4]. The first step in the understanding of such effects is simply to undertake the study of quantum Landau systems. So, in this paper we revise the Landau problem from the symmetry optics.

The relevant symmetry group of the magnetic field in the planar Landau system is the Euclidean group $E(2)$. In the same way, the associated symmetry groups of the spherical and hyperbolic systems are $SO(3)$ and $SO(2,1)$, respectively. Moreover, the configuration spaces of such Landau systems (i.e. sphere, plane and hyperboloid) can be seen as homogeneous spaces of their corresponding symmetry groups. Thus, we will set up the following project: (i) to carry out a simultaneous study of these classes of Landau systems by using a unifying formalism that will allow us to compare directly the features of all of them (in some sense, a study of these Landau systems sharing our point of view was done in [5]); (ii) to clearly characterize the elements of the Landau systems that can be explained exclusively in terms of group theoretical arguments.

We shall develop the first point of this program thoroughly starting from the definition of a general symmetry group up to the final solutions of the wave equations. In particular, we will understand the correct way in which the planar Landau quantum system can be seen as a limit of the spherical or hyperbolic systems when the surface curvature vanishes. This question deserves careful attention because it displays how trivial extensions can lead to a nontrivial one, much in the same way as the Poincaré group leads to the extended Galilei group (which is essential in describing the mass of nonrelativistic systems).

With respect to the second point, up to now the Landau systems were defined by means of Schrödinger equations, and their symmetries played a complementary role as a help to solve the spectrum. Now, in our viewpoint the key object is the symmetry group itself from which to develop a certain canonical procedure to get the quantum Landau systems. We will see that the main clue in dealing with this problem is the concept of locality. Thus, as Bargmann and Wigner already stressed [6], local representations (or locally operating representations) of Lie groups of space–time transformations [7] constitute a relevant ingredient in QM. Here, we shall show that local representations of the symmetry groups are the right approach to describe the Landau systems (as it was done with the Euclidean group [7,8] or with the Maxwell groups in [9]) providing us at the same time with the minimal coupling rule of interaction with the external magnetic field. In conclusion, we can state that, from the symmetry point of view, local equivalence is responsible for the classification of different Landau levels defined on any surface. To show the way this is realized, and its physical implications, will be one of the main objectives of this work.

The natural framework to write down local representations is the language of fibre bundles, so we shall briefly consider this point in our exposition, but leave the technical details to the quoted references in order to shorten the length of this paper.

The organization of this paper is as follows. Section 2 introduces a general group (in fact a one-parameter family of groups) that includes the three symmetry groups mentioned above, together with their homogeneous spaces. We also consider the central extensions of such groups which we will call ‘magnetic Landau’ groups. In section 3 we characterize the local representations of this general group that will be relevant to define the Schrödinger wave equations for quantum systems supporting this symmetry group in section 4. Some basic facts related to the formulation of gauge invariant potentials under local realizations in the framework of fibre bundles are presented in section 5. They will allow us to give a group-theoretical justification of the minimal electromagnetic coupling. In section 6 we classify the elementary systems associated with the magnetic Landau groups in the sense of Wigner [10], i.e. an elementary quantum system is associated with a unitary irreducible realization of the symmetry group (here we will restrict ourselves to bounded representations). Afterwards, we decompose the local representations of the magnetic Landau groups in terms of their elementary systems in order to get the energy spectrum and eigenfunctions of the corresponding Landau quantum systems. In section 7 we present the variable separation of the hyperbolic Landau system using the horocyclic coordinates of $SO(2,1)$. In this way we reduce the quantum Landau problem to a system of a particle moving in a Morse potential allowing one to understand the continuous spectrum of the hyperbolic system (this question was previously addressed but only at a classical level). In section 8 we construct ladder operators connecting eigenstates of consecutive eigenvalues of the spectrum (for $\kappa \neq 0$ such operators have not been considered previously up to our knowledge). These ladder operators have some interesting properties: (i) they satisfy essentially cubic commutation relations; (ii) they connect the Landau systems to isotropic oscillators on constant curvature surfaces; and (iii) they allow one to derive directly the spectrum even when the wavefunctions obey different boundary conditions (this is the case of the 'moving states' discussed in [5]). Finally, section 9 displays the main results in a more physical language together with some general remarks and comments. Some appendices have been added in order to make the work as self-contained as possible: appendix A gives a short review of local realizations; appendix B characterizes the local representations of the magnetic Landau groups and appendix C supplies some basic elements of fibre bundles and gauge theories.

2. Symmetry groups of Landau quantum systems

The first step to achieve our program is to propose a unified notation by introducing a Lie group, denoted by $SO_\kappa(3)$ [11], involving the three aforementioned symmetry groups, and a homogeneous space which also includes as particular cases the three types of 2D surfaces where the quantum Landau systems live.

2.1. Symmetry groups of constant magnetic fields

As we mentioned in the introduction the suitable symmetry groups of our Landau systems are $SO(3)$, $E(2)$ and $SO(2,1)$. They can be dealt with in a more compact way by defining a one-parameter family of Lie groups $SO_\kappa(3)$, with κ a real parameter, whose Lie algebra, $so_\kappa(3)$, is generated by the infinitesimal (Hermitian) generators J_{01}, J_{02} and J_{12} satisfying the following Lie commutators:

$$[J_{01}, J_{02}] = \mathrm{i}\kappa J_{12} \qquad [J_{12}, J_{01}] = \mathrm{i}J_{02} \qquad [J_{12}, J_{02}] = -\mathrm{i}J_{01}. \tag{2.1}$$

When κ is nonzero it can be rescaled to +1 or −1, whence we have three representative values: +1, 0, −1. If $\kappa = +1$ we recover the Lie algebra $so(3)$; for $\kappa = 0$ we have the Lie algebra $e(2)$ of the 2D Euclidean group $E(2)$; and finally, when $\kappa = -1$, we get $so(2,1)$. The quadratic Casimir of $so_\kappa(3)$ is $C_\kappa = J_{01}^2 + J_{02}^2 + \kappa J_{12}^2$.

The group $SO_\kappa(3)$ admits a linear action in the ambient space $\mathbb{R}^3$, leaving invariant the quadratic form $\langle x, x\rangle_\kappa = x_0^2 + \kappa x_1^2 + \kappa x_2^2$, $x \in \mathbb{R}^3$. The matrix representation (that explains the index notation) of the generators is

$$J_{01} = \mathrm{i}\,(-\kappa E_{01} + E_{10}) \qquad J_{02} = \mathrm{i}\,(-\kappa E_{02} + E_{20}) \qquad J_{12} = \mathrm{i}\,(-E_{12} + E_{21}) \tag{2.2}$$

where the 3×3 matrices E_{ij} are defined by $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, $i, j, k, l = 0, 1, 2$. In this representation, the orbit of the point $x_0 = (1, 0, 0)$ is the 2D surface S_κ^2 of equation $x_0^2 + \kappa x_1^2 + \kappa x_2^2 = 1$. This surface is diffeomorphic to the homogeneous space $SO_\kappa(3)/SO(2)$, where $SO(2)$ is the isotropy group of x_0 spanned by J_{12} (the only compact generator of $so_\kappa(3)$ for every value of κ). For $\kappa = +1, 0, -1$, S^2_κ is the 2-sphere, S^2, the Euclidean plane, E^2, and the hyperboloid, H^2, respectively. So, the parameter κ appearing in the commutation rules (2.1) can also be interpreted as the curvature of S^2_κ. In particular, if $\kappa = 0$ the metric $\langle x, x\rangle_\kappa$ is degenerate and the homogeneous space is flat (for more details see [12]).

The contraction process, that allows one to obtain $E(2)$ from $SO(3)$ or $SO(2,1)$, is equivalent in our framework to simply take $\kappa = 0$ in (2.1). This replacement can be interpreted geometrically as a deformation where the curvature radius $R = 1/\sqrt{\kappa}$ ($R = 1/\sqrt{-\kappa}$ for the hyperboloid) goes to ∞. In this way the Euclidean plane becomes the limit of the sphere or the hyperboloid.

A useful chart of S^2_κ is given by polar geodesic coordinates. Let us consider again the point $x_0 = (1, 0, 0)$ of S^2_κ, then any other point x of S^2_κ is parametrized by the pair (r, θ) according to the following action of $SO_\kappa(3)$:

$$x = \mathrm{e}^{-\mathrm{i}\theta J_{12}}\mathrm{e}^{-\mathrm{i}r J_{01}}x_0. \tag{2.3}$$

If κ is positive, $(r, \theta) \in (0, \pi/\sqrt{\kappa}) \times (0, 2\pi)$, while for κ zero or negative $(r, \theta) \in (0, \infty) \times (0, 2\pi)$. So, this chart covers S^2_κ except the two 'poles' (taking the point x_0 as the 'north pole' and placing the 'south pole' at infinity for the non-compact cases, E^2 and H^2) and the meridian joining them. The explicit expression of this coordinate system is

$$x^0 = \cos\sqrt{\kappa}r \qquad x^1 = \sin\sqrt{\kappa}r\cos\theta/\sqrt{\kappa} \qquad x^2 = \sin\sqrt{\kappa}r\sin\theta/\sqrt{\kappa}. \tag{2.4}$$

With this convention, the contracted 2D plane S^2_κ in the limit $\kappa \to 0$ is given by $x^0 = 1$, that is, we have chosen the contraction around the north pole $x_0 = (1, 0, 0)$.

The fundamental vector fields associated with the basis generators of $so_\kappa(3)$ that correspond to the action of $SO_\kappa(3)$ on S^2_κ are

$$\begin{aligned} J_{01}(r,\theta) &= -\mathrm{i}\cos\theta\,\partial_r + \mathrm{i}\sqrt{\kappa}\,\frac{\sin\theta}{\tan\sqrt{\kappa}r}\,\partial_\theta \\ J_{02}(r,\theta) &= -\mathrm{i}\sin\theta\,\partial_r - \mathrm{i}\sqrt{\kappa}\,\frac{\cos\theta}{\tan\sqrt{\kappa}r}\,\partial_\theta \\ J_{12}(r,\theta) &= -\mathrm{i}\,\partial_\theta. \end{aligned} \tag{2.5}$$

These formulae are valid for any value of κ. Note that for $\kappa < 0$ we have hyperbolic functions, while for $\kappa = 0$

$$\lim_{k\to 0}\cos\sqrt{\kappa}r = 1 \qquad \lim_{k\to 0}\frac{\sin\sqrt{\kappa}r}{\sqrt{\kappa}} = r.$$

Hence, when $\kappa = 1, -1$ or 0 expressions (2.5) give the usual vector fields of $so(3)$, $so(2, 1)$ or $e(2)$, respectively. In particular, for $\kappa = 0$ we immediately obtain the Euclidean fields on the plane:

$$\begin{aligned} &J_{01}(r,\theta) = -\mathrm{i}\cos\theta\,\partial_r + \mathrm{i}\frac{\sin\theta}{r}\,\partial_\theta \qquad J_{02}(r,\theta) = -\mathrm{i}\sin\theta\,\partial_r - \mathrm{i}\frac{\cos\theta}{r}\,\partial_\theta \\ &J_{12}(r,\theta) = -\mathrm{i}\partial_\theta. \end{aligned}$$

Notice that in the Euclidean limit J_{01} and J_{02} become the generators of translations along the Cartesian axes X and Y respectively, while J_{12} corresponds to the generator of rotations with respect to the Z-axis; in this case they are usually denoted by P_1, P_2 and J.

The invariant measure in S^2_κ is given, up to a constant factor, by

$$\sigma = \frac{\sin\sqrt{\kappa}r}{\sqrt{\kappa}}\,\mathrm{d}r \wedge \mathrm{d}\theta. \tag{2.6}$$

In the limit $\kappa \to 0$ we recover the usual Euclidean measure $\sigma = r\,\mathrm{d}r \wedge \mathrm{d}\theta$.

There are other (group) coordinates (for instance, parallel geodesic or horocyclic) that would be of interest to analyse particular aspects. However, polar geodesic coordinates are more suitable to handle bases of eigenfunctions of J_{12} for which the realization (2.5) is well adapted.

2.2. *Magnetic Landau groups*

If a physical system has a symmetry group G, in QM its symmetry transformations are described by projective representations in the space of rays, or by representations up to a factor in the associated Hilbert space [10, 13]. Such representations can be obtained by means of true representations of an extended group $\overline{G}$ that Wigner called a 'quantum mechanical symmetry group' (appendix A).

In our case (see appendix B) $\overline{G}$ is a central extension of (the universal covering of) $SO_\kappa(3)$ by $\mathbb{R}$ which will be denoted $\overline{SO}_\kappa(3)$ and in the following it will be referred to as the family of 'magnetic Landau groups'. The basis $\{\overline{J}_{01}, \overline{J}_{02}, \overline{J}_{12}, B\}$ of $\overline{so}_\kappa(3)$, the Lie algebra of $\overline{SO}_\kappa(3)$, includes a new generator B corresponding to the central extension. The commutators of $\overline{so}_\kappa(3)$ are given by

$$\begin{array}{ll} [\overline{J}_{01}, \overline{J}_{02}] = \mathrm{i}\,\kappa\,\overline{J}_{12} + \mathrm{i}\,B, & [\overline{J}_{12}, \overline{J}_{01}] = \mathrm{i}\,\overline{J}_{02}, \\ [\overline{J}_{12}, \overline{J}_{02}] = -\mathrm{i}\,\overline{J}_{01}, & [., B] = 0. \end{array} \tag{2.7}$$

From (2.7) it is easy to see at the level of Lie algebras that only when $\kappa = 0$ the extension is nontrivial, giving in this case the extended Euclidean algebra $\overline{e}(2)$ [7].

The group law of $\overline{SO}_\kappa(3)$ can be obtained from the Lie algebra (2.7), but we shall never need it; for us it will be enough to work with the infinitesimal generators bearing in mind its physical meaning. The second-order Casimir is

$$\overline{C}_\kappa = \overline{J}_{01}^2 + \overline{J}_{02}^2 + \kappa\,\overline{J}_{12}^2 + 2B\overline{J}_{12}. \tag{2.8}$$

The homogeneous space S_κ^2 can also be expressed as $S_\kappa^2 \approx SO_\kappa(3)/SO(2) = \overline{SO}_\kappa(3)/(\overline{SO}(2) \otimes \mathbb{R})$, where $\overline{SO}(2)$ is a covering (depending on κ) of $SO(2)$, and $\mathbb{R}$ is the group generated by B. Since the extension is central, the action of the subgroup $\langle B\rangle$ on S_κ^2 is trivial.

3. Local representations of the magnetic Landau groups

In appendix A the reader can find a brief review of the theory of local representations and in appendix B we show how to build up the local representations of the magnetic groups $\overline{SO}_\kappa(3)$, which are the suitable ones to describe the quantum symmetries of $SO_\kappa(3)$. We shall present in the following the results necessary for our development.

The local representations (A.1) of the basis generators of $\overline{so}_\kappa(3)$ are given by Hermitian differential operators that have the general form

$$\overline{X}_j(x) = X_j(x) + W_j(x) \qquad B = -\beta \tag{3.1}$$

where $\overline{X}_j \in \{\overline{J}_{01}, \overline{J}_{02}, \overline{J}_{12}\}$; $X_j(x)$ are the fundamental fields (2.5), $W_j(x)$ are real functions, and β is a real number that represents the central generator B and specifies the factor system of the realization. The final explicit expressions (obtained according to appendix B) for the infinitesimal generators (3.1), using polar coordinates (2.4), are

$$
\begin{aligned}
\overline{J}_{01} &= J_{01}(r,\theta) - \beta\,\mathrm{vers}_\kappa r\,\frac{\sqrt{\kappa}\sin\theta}{\sin\sqrt{\kappa}\,r}\\
\overline{J}_{02} &= J_{02}(r,\theta) + \beta\,\mathrm{vers}_\kappa r\,\frac{\sqrt{\kappa}\cos\theta}{\sin\sqrt{\kappa}\,r}\\
\overline{J}_{12} &= J_{12}(r,\theta)\\
B &= -\beta
\end{aligned}
\tag{3.2}
$$

where the fields $J_{..}(r,\theta)$ are given in (2.5). We have also introduced a general versine function $\mathrm{vers}_\kappa r = 1/\kappa(1-\cos\sqrt{\kappa}r)$ that has a well defined limit, $\lim_{\kappa\to 0}\mathrm{vers}_\kappa r = r^2/2$.

We shall remark upon some of the important features of the above realization (3.2). (i) First of all, it is instructive to check that expressions (3.2) indeed satisfy the commutation rules (2.7). (ii) The (extended) fields (3.2) are smooth around the north pole x_0, so that they act on functions also differentiable there. (iii) The main point to stress here is that, as it is detailed in appendix B, each class of local equivalence for the extended fields of the form (3.1) satisfying (2.7) is characterized by β, where $\beta\in\mathbb{R}$ if $\kappa\leqslant 0$, or $2\beta/\kappa\in\mathbb{Z}$ if $\kappa>0$. We will assume henceforth that $2\beta/\kappa\in\mathbb{Z}$, which is valid for all values of κ. The reason underlying the discretization of β is the same as with respect to the spin: only half-integer values are allowed in the (projective) representations of $SO(3)$. Other values of β would lead us to a representation of the algebra, not of the group.

The fields (3.2), defined up to a local equivalence, determine a trivial extension for $\kappa\neq 0$. When $\kappa\to 0$ the extension becomes nontrivial. Following the arguments of appendix B, the limit $\kappa\to 0$ of (3.2) must be performed bearing in mind that $2\beta/\kappa\in\mathbb{Z}$. If we keep $\beta=\beta_0$ fixed, this contraction is discrete since $\kappa = 2\beta_0/n$, $n\in\mathbb{N}$, and $n\to\infty$.

4. Schrödinger equations for Landau systems

Once we obtain the local realizations of $SO_\kappa(3)$, we can characterize the quantum elementary systems behaving under this type of symmetry transformations. Thus, we will assume that the support space of the local realization contains the Hilbert space of wavefunctions of the system. By using the invariant measure (2.6) and restricting ourselves to square integrable functions, we obtain the physical states. The infinitesimal generators of the symmetry group must have a Hermitian character in order to be identified as observables of the system; in other words, we must consider unitary representations. Finally, the time evolution is given by a Schrödinger equation $\mathrm{i}\partial_t\Psi = H_\kappa\Psi$, where the Hamiltonian we are going to consider is essentially the Casimir (2.8), $H_\kappa = \overline{C}_\kappa/2$ (it can be redefined up to additive or multiplicative constants). Its explicit expression after substituting in (2.8) the generators by their associated vector fields (3.2) is

$$
\begin{aligned}
H_\kappa = {}& -\frac{1}{2}\,\partial_r^2 - \frac{\kappa}{2\sin^2\sqrt{\kappa}r}\,\partial_\theta^2 + \mathrm{i}\left(\kappa\,\beta\,\mathrm{vers}_\kappa r\,\frac{\cos\sqrt{\kappa}r}{\sin^2\sqrt{\kappa}r} + \beta\right)\partial_\theta\\
& - \frac{\sqrt{\kappa}}{2\tan\sqrt{\kappa}r}\,\partial_r + \frac{\kappa}{2\sin^2\sqrt{\kappa}r}\,(\beta\,\mathrm{vers}_\kappa r)^2.
\end{aligned}
\tag{4.1}
$$

In general, the local representations are reducible, each irreducible component is given by the Casimir equation $\overline{C}_\kappa\Psi = \overline{c}_\kappa\Psi$. Whence, by construction, each eigenspace of H_κ supports a unitary irreducible representation (UIR) of $\overline{SO}_\kappa(3)$, since the eigenvalue equation $H_\kappa\Psi_\kappa = \varepsilon_\kappa\Psi_\kappa$, $\varepsilon_\kappa = \overline{c}_\kappa/2$, gives the irreducible subspaces of the local representation. The description of our quantum system will be complete if we compute the spectrum, the degeneracy of the energy levels (given by the aforementioned UIR) and a set of orthogonal eigenfunctions generating the full Hilbert space of states.

5. Gauge potentials and minimal coupling

In section 3 we introduced the local realizations of $SO_\kappa(3)$ in a direct operative way often used in the physics literature. However, as we mentioned in section 1, the natural framework for the local realizations is the fibre bundle theory. We shall analyse, in this section, some properties obtained from this more general viewpoint that allows us to interpret physically (and geometrically) what is behind the Hamiltonian (4.1) that we proposed in the preceding section, and it will also help us to derive the minimal coupling rule for interactions. For more details see appendix C.

5.1. Gauge invariant potentials

We can find a gauge invariant potential $A_\mu(x)$ under the action (3.2) of $\overline{SO}_\kappa(3)$. The local invariance condition of the potential gives the following set of differential equations:

$$X_j^\mu(x)\frac{\partial A_\nu(x)}{\partial x^\mu} + A_\mu(x)\frac{X_j^\mu(x)}{\partial x^\nu} - \mathrm{i}\frac{\partial W_j(x)}{\partial x^\nu} = 0 \qquad \mu,\nu = 1,2, \forall X_j \in \overline{so}_\kappa(3) \tag{5.1}$$

where the fields $X_j(x)$, and the functions $W_j(x)$ of the local realization were defined in (3.1) and (3.2). It can be shown that this potential is the pull-back of a global invariant connection defined on a $U(1)$ principal bundle whose base space is S^2_κ.

The solutions to equation (5.1), taking coordinates $x^1 = r$ and $x^2 = \theta$, are

$$A_r = 0 \qquad A_\theta = \beta\,\mathrm{vers}_\kappa r. \tag{5.2}$$

Such a solution is differentiable in a chart covering S^2_κ, except for the south pole (as it was foreseeable, since the local realization was smooth there). This is the appropriate chart for our contraction around the north pole. As usual, we can define the covariant derivatives by

$$D_r = -\mathrm{i}\partial_r - A_r \qquad D_\theta = -\mathrm{i}\partial_\theta - A_\theta. \tag{5.3}$$

Thus, the component of the invariant curvature form is

$$B_{r\theta} = -\mathrm{i}\,[D_r, D_\theta] = \frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}}\,\beta \tag{5.4}$$

which corresponds to a magnetic field normal to S^2_κ whose intensity is given by β (recall the invariant measure (2.6)). Remark that if $\kappa \neq 0$, we have taken $2\beta/\kappa \in \mathbb{Z}$, which in the case $\kappa = 1$ coincides with the Dirac monopole quantization [14]. If we want this intensity to be conserved along the limiting process $\kappa \to 0$ we must take $\beta(\kappa) = \beta_0$, i.e. a constant independent of κ. With this choice the potential (5.2) has a well defined limit: $A_r = 0$, $A_\theta = \beta_0 r^2/2$.

5.2. Minimal coupling interactions

Now we shall see how the minimal coupling rule can be introduced using arguments based on the symmetry algebra.

Let $\{X_i = X_i^\mu(x)\partial_\mu\}$ be the vector field realization (2.5) on the pseudosphere S^2_κ of the Lie algebra basis of $so_\kappa(3)$, and let us consider the new set of generators $X_i^* = X_i^\mu(x)D_\mu$, with D_μ the covariant derivative (5.3). As we stated above, the Casimir operator $\overline{C}_\kappa(\overline{X}_i, B)$ of $\overline{so}_\kappa(3)$ was identified (up to the factor 1/2) with the Hamiltonian of our quantum system. Now, according to expression (C.3) of appendix C, this operator can be obtained from the Casimir

$C_\kappa(X_i)$ of $so_\kappa(3)$ substituting the fields X_i by X_i^*: $\overline{C}_\kappa(\overline{X}_i, B) = C_\kappa(X_i^*)$. Making use of this property we can rewrite the Hamiltonian (4.1) in terms of the vector fields X_i^* as

$$\begin{aligned} H_\kappa &= -\frac{1}{2} D_r^2 - \frac{\kappa}{\sin^2\sqrt{\kappa}r} D_\theta^2 - \frac{1}{2}\frac{\sqrt{\kappa}}{\tan\sqrt{\kappa}r} D_r \\ &= -\frac{1}{2}\partial_r^2 + \frac{1}{2}\frac{\kappa}{\sin^2\sqrt{\kappa}r}(-\mathrm{i}\,\partial_\theta - \beta\,\mathrm{vers}_\kappa r)^2 - \frac{1}{2}\frac{\sqrt{\kappa}}{\tan\sqrt{\kappa}r}\partial_r. \end{aligned} \tag{5.5}$$

The advantage of (5.5) is that it makes explicit the minimal coupling rule since it is the Hamiltonian of a free system on S_κ^2 where the derivatives have been replaced by covariant derivatives. Therefore, (5.5) describes the interaction of a quantum system with an external magnetic field (5.4) normal to the surface S_κ^2 given by the electromagnetic potential (5.2).

The limit $\kappa \to 0$ of the time-independent Schrödinger equation $H_\kappa\Psi_\kappa = \varepsilon_\kappa\Psi_\kappa$ is

$$-\frac{1}{2}\partial_r^2\Psi(r,\theta) + \frac{1}{2r^2}\left(-\mathrm{i}\,\partial_\theta - \beta\frac{r^2}{2}\right)^2\Psi(r,\theta) - \frac{1}{2r}\partial_r\Psi(r,\theta) = \varepsilon_0\Psi(r,\theta). \tag{5.6}$$

This is the eigenvalue equation in polar coordinates of a charged particle in the Euclidean plane under the action of a constant magnetic field of intensity proportional to $|\beta|$ perpendicular to the plane, that is, the planar Landau system [1].

6. Eigenvalues and eigenfunctions of Landau Hamiltonians

As a first step to obtaining the spectrum and eigenfunctions of the Hamiltonian (5.5) we will compute the UIRs of $\overline{SO}_\kappa(3)$; afterwards, we will analyse their relationship with the local representations determined in section 3.

6.1. Elementary Landau quantum systems

A basis of an UIR of $\overline{SO}_\kappa(3)$ is completely characterized by the eigenvalues and eigenvectors of three mutually commuting operators: H_κ $(= 1/2\,\overline{C}_\kappa)$, B and $\overline{J}_{12}$. Let us denote by $|\varepsilon_\kappa, \beta, m\rangle$ an eigenvector of these operators, i.e.,

$$H_\kappa|\varepsilon_\kappa,\beta,m\rangle = \varepsilon_\kappa|\varepsilon_\kappa,\beta,m\rangle \qquad \overline{J}_{12}|\varepsilon_\kappa,\beta,m\rangle = m|\varepsilon_\kappa,\beta,m\rangle$$
$$B|\varepsilon_\kappa,\beta,m\rangle = -\beta|\varepsilon_\kappa,\beta,m\rangle.$$

The UIRs of the universal covering of $\overline{SO}_\kappa(3)$ can be obtained from those of $\overline{so}_\kappa(3)$. We look for expressions valid for any κ-value and at the same time for bounded representations. However, for $\kappa < 0$ the group is noncompact and there are other unitary representations (the principal and complementary series) not considered here.

Let $\{J^+, J^-, J, B\}$ be a 'Cartan' basis of $\overline{so}_\kappa(3)$, where $J^\pm = 1/\sqrt{2}(\overline{J}_{01} \pm \mathrm{i}\overline{J}_{02})$ and $J = \overline{J}_{12}$. The non-vanishing Lie brackets are now

$$[J, J^\pm] = \pm J^\pm \qquad [J^+, J^-] = \kappa J + B. \tag{6.1}$$

Since the representation must be unitary, then the generators must satisfy the Hermitian relations $(J^\pm)^\dagger = J^\mp$, $J^\dagger = J$, $B^\dagger = B$. The second-order Casimir in the new basis reads

$$\overline{C}_\kappa = 2J^+J^- + \kappa(J^2 - J) + 2BJ - B = 2J^-J^+ + \kappa(J^2 + J) + 2BJ + B. \tag{6.2}$$

There are two families of such bounded UIRs of $\overline{SO}_\kappa(3)$ characterized as follows. One of them is given by a lowest negative weight, $-l$ $(l \in \mathbb{Z}^+)$, such that $J^-|\varepsilon_\kappa, \beta, -l\rangle = 0$, and the other one by a highest positive weight, $l \in \mathbb{Z}^+$, verifying $J^+|\varepsilon_\kappa, \beta, l\rangle = 0$.

For the first family of UIRs the action of the generators $J^{\pm}$ on the states $|\varepsilon_\kappa, \beta, m\rangle$ can be written as

$$
\begin{aligned}
J^{+}|\varepsilon_\kappa, \beta, m\rangle &= \sqrt{(l+m+1)(2\beta+\kappa(l-m))/2}\,|\varepsilon_\kappa, \beta, m+1\rangle \\
J^{-}|\varepsilon_\kappa, \beta, m\rangle &= \sqrt{(l+m)(2\beta+\kappa(l-m+1))/2}\,|\varepsilon_\kappa, \beta, m-1\rangle
\end{aligned}
\tag{6.3}
$$

with the restriction

$$(l+m)(2\beta+\kappa(l-m+1)) \geqslant 0. \tag{6.4}$$

This bounded representation is determined by the eigenvalue of the Casimir (6.2), labelled by the integer l,

$$\varepsilon_\kappa^l = \kappa l(l+1)/2 + \beta(l+1/2). \tag{6.5}$$

The features of the UIRs of the first family depending on the particular values of κ can be summarized as follows:

- $\kappa > 0$: $l \in \mathbb{Z}^{\geqslant 0}$, $2\beta/\kappa \in \mathbb{Z}$ such that $-l < \beta/\kappa$. The representation has dimension $2(l+\beta/\kappa)+1$ with carrier space generated by the set of eigenvectors

$$\{|\varepsilon_\kappa^l, \beta, m\rangle\}_{m=-l}^{l+2\beta/\kappa}. \tag{6.6}$$

Therefore, there are an infinite number of levels, each one finitely degenerated.
- $\kappa < 0$: $l \in \mathbb{Z}^{\geqslant 0}$, $2\beta/\kappa \in \mathbb{R}$ and $l < \beta/|\kappa|$. We obtain an infinite-D representation with support space spanned by

$$\{|\varepsilon_\kappa^l, \beta, m\rangle\}_{m=-l}^{\infty}. \tag{6.7}$$

There is a finite number of discrete energy levels, $0 \leqslant l < \beta/|\kappa|$, each one infinitely degenerated.
- $\kappa = 0$: $l \in \mathbb{Z}^{\geqslant 0}$ and $\beta > 0$. The representation is infinite-D with a basis of the support space given by

$$\{|\varepsilon_\kappa^l, \beta, m\rangle\}_{m=-l}^{\infty}. \tag{6.8}$$

In this case we have infinite discrete energy levels infinitely degenerated.

In order to take into consideration the three cases together we will assume that $\beta > 0$. In all of them ε_κ^l is given by (6.5).

For the second family of UIRs we have

$$
\begin{aligned}
J^{+}|\varepsilon_\kappa, \beta, m\rangle &= \sqrt{(l-m)(-2\beta+\kappa(l+m+1))/2}\,|\varepsilon_\kappa, \beta, m+1\rangle \\
J^{-}|\varepsilon_\kappa, \beta, m\rangle &= \sqrt{(l-m-1)(-2\beta+\kappa(l+m))/2}\,|\varepsilon_\kappa, \beta, m-1\rangle
\end{aligned}
\tag{6.9}
$$

with the restriction $(l-m)(-2\beta+\kappa(l+m+1)) \geqslant 0$. A bounded representation is determined by the eigenvalue of the Casimir (6.2), labelled by the positive integer l,

$$\varepsilon_\kappa^l = \kappa l(l+1)/2 - \beta(l+1/2). \tag{6.10}$$

Similarly to the first family we have the three following cases according with the values of κ:

- $\kappa > 0$: $l \in \mathbb{Z}^{\geqslant 0}$, $2\beta/\kappa \in \mathbb{Z}$, such that $l > \beta/\kappa$. Basis: $\{|\varepsilon_\kappa^l, \beta, m\rangle\}_{-l+2\beta/\kappa}^{m=l}$.
- $\kappa < 0$: $l \in \mathbb{Z}^{\geqslant 0}$, $2\beta/\kappa \in \mathbb{Z}$, $-\beta > l|\kappa|$. Basis: $\{|\varepsilon_\kappa^l, \beta, m\rangle\}_{-\infty}^{m=l}$.
- $\kappa = 0$: $l \in \mathbb{Z}^{\geqslant 0}$, $\beta < 0$. Basis: $\{|\varepsilon_\kappa^l, \beta, m\rangle\}_{-\infty}^{m=l}$.

Now it is appropriate to consider $\beta < 0$ for the three cases, while ε_κ^l is given by (6.10). The same comments about dimensionality and degeneration of the Landau levels made for the other family of UIRs are also valid in this case.

At this point we can check that the contraction process works correctly for the UIRs defined above. For instance, the finite-D representations of $\overline{SO}(3)$ contract to infinite-D ones of the 2D Euclidean group provided that $\kappa = 2|\beta|/n \to 0$, $n \in \mathbb{N}$, when $n \to \infty$ (for more details see [15]).

It is worth remarking that the energy eigenvalues (6.5) and (6.10) include two terms. The first of them is quadratic in l and has a geometric character through the curvature κ of the configuration space. The second term, linear in l, is the only one that will remain in the planar limit and has a dynamic character by means of β that was interpreted as a magnetic field.

6.2. *A complete set of eigenfunctions*

Once the above UIRs of $\overline{SO}_\kappa(3)$ have been characterized we have to check whether they are realizable as irreducible components of the local representations given in section 3.

Recall that we must restrict ourselves to differentiable wavefunctions around the north pole. So, we can write $\Psi_{l,m}^{\beta,\kappa}(r,\theta) = \mathrm{e}^{\mathrm{i}m\theta} R_{l,m}^{\beta,\kappa}(r)$, $m \in \mathbb{Z}$, as the wavefunction associated with the basis element $|\varepsilon_\kappa, \beta, m\rangle$, i.e. $\langle r,\theta|\varepsilon_\kappa,\beta,m\rangle = \Psi_{l,m}^{\beta,\kappa}(r,\theta)$, of a lowest weight representation with Casimir eigenvalue (6.5) given by $\overline{c}_\kappa = \varepsilon_\kappa^l/2$. From (3.2) the local expressions of the up and down operators take the form

$$J^\pm = \mathrm{i}\,\mathrm{e}^{\pm\mathrm{i}\theta}\left(-\partial_r \mp \frac{\mathrm{i}\sqrt{\kappa}}{\tan\sqrt{\kappa}r}\partial_\theta \pm \beta\,\mathrm{ver}_\kappa(r)\frac{\sqrt{\kappa}}{\sin\sqrt{\kappa}r}\right). \tag{6.11}$$

For each eigenvalue, m, of $J \equiv J_{12}$ we can compute the radial eigenfunctions $R_{l,m}^{\beta,\kappa}(r)$ quite easily. So, the fundamental wavefunction corresponding to $|\varepsilon_\kappa, \beta, -l\rangle$ is determined by the equation

$$\left(-\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l\sqrt{\kappa}}{\tan\sqrt{\kappa}r} - \beta\,\mathrm{ver}_\kappa(r)\frac{\sqrt{\kappa}}{\sin\sqrt{\kappa}r}\right) R_{l,-l}^{\beta,\kappa}(r) = 0 \tag{6.12}$$

whose solution (up to normalization) is

$$R_{l,-l}^{\beta,\kappa}(r) = \left(\frac{\sin\sqrt{\kappa}r}{\sqrt{\kappa}}\right)^l \left(1 + \tan^2\frac{\sqrt{\kappa}r}{2}\right)^{-\beta/\kappa}. \tag{6.13}$$

Once chosen $\beta > 0$, the complete function $\Psi_{l,-l}^{\beta,\kappa}(r,\theta) = \mathrm{e}^{-\mathrm{i}l\theta} R_{l,-l}^{\beta,\kappa}(r)$ is square integrable on the 'sphere' S_κ^2 with respect to the invariant measure (2.6) for $\kappa \geqslant 0$, or if $0 \leqslant l < \beta/|\kappa| - 1/2$ for $\kappa < 0$. From this eigenfunction and using the raising operator J^+ we can find all the remaining basis eigenfunctions generating the whole ε_κ^l-eigenspace:

$$\Psi_{l,-l+n}^{\beta,\kappa}(r,\theta) \propto (J^+)^n\,\Psi_{l,-l}^{\beta,\kappa}(r,\theta). \tag{6.14}$$

These eigenfunctions are also square integrable provided the requirements (6.6), (6.7) of the previous section are fulfilled besides $l < \beta/|\kappa| - 1/2$ when $\kappa < 0$. In this way we have completed the search for the spectrum and eigenfunctions of the Hamiltonian (5.5) for any value of κ.

Remark that for $\kappa = 0$ the Landau energy levels (6.5) are linear in l, $\varepsilon_0^l = 2\beta\,(l + 1/2)$. The fundamental state inside the ε_0^l-eigenspace is obtained by taking the limit $\kappa \to 0$ of (6.13),

$$R_{l,m}^{\beta,0}(r) = r^l \mathrm{e}^{-\beta r^2/4}. \tag{6.15}$$

This can be used to derive the rest of the infinite basis eigenfunctions with the help of the shift operators

$$J^{\pm} = \mathrm{i}\,\mathrm{e}^{\pm\mathrm{i}\theta}\left(-\partial_r \mp \frac{\mathrm{i}}{r}\partial_\theta \pm \frac{\beta r}{2}\right).$$

The results for the second family of UIRs are quite similar taking into account obvious sign changes in m and β (recall in this respect that now β is negative). For instance, the fundamental state is $\Psi^{\beta,\kappa}_{l,l}(r,\theta) = \mathrm{e}^{\mathrm{i}\,l\theta} R^{\beta,\kappa}_{l,l}(r)$, where

$$R^{\beta,\kappa}_{l,l}(r) = \left(\frac{\sin\sqrt{\kappa}r}{\sqrt{\kappa}}\right)^{l}\left(1+\tan^2\frac{\sqrt{\kappa}r}{2}\right)^{\beta/\kappa}. \tag{6.16}$$

6.3. Lowest Landau level

A case of special interest is when $l = 0$. Once the geometry (i.e. κ) and the external field (β) are fixed this value corresponds to the lowest energy Landau level. From (6.14) it is easy to show that the (radial component of the) eigenfunctions are simply given by

$$R^{\beta,\kappa}_{0,m}(r) = N(\beta,\kappa,m)\,(\cos\sqrt{\kappa}r/2)^{2\beta/\kappa-m}\left(\frac{\sin\sqrt{\kappa}r/2}{\sqrt{\kappa}}\right)^{m} \tag{6.17}$$

with the normalizing coefficient $N(\beta,\kappa,m) = (2(2\beta+\kappa)\cdots(2\beta+\kappa-m\kappa)/\Gamma(m+1))^{1/2}$. If we further select $\kappa = 0$ the above formulae reduce to the lowest level of the planar Landau system, whose eigenfunctions are

$$R^{\beta,0}_{0,m}(r) = N(\beta,0,m)\,(r/2)^m\,\mathrm{e}^{-\beta r^2/4} \tag{6.18}$$

with $N(\beta,0,m) = (2^m\beta^{m+1}/\,\Gamma(m+1))^{1/2}$.

The state density (degeneracy of the lth level/area) for the spherical case is

$$\frac{2l+1+2|\beta|\kappa}{4\pi/k} = \frac{|\beta|}{2\pi} + \frac{\kappa(2l+1)}{4\pi}. \tag{6.19}$$

In the Euclidean limit ($\kappa = 0$) the state density is $|\beta|/2\pi$ for any l in agreement with the Landau result [1]. In the hyperbolic case the state density is also $|\beta|/2\pi$.

6.4. Eigenfunctions in terms of hypergeometric functions

The components $R^{\beta,\kappa}_{l,m}(r)$ of the basis wavefunctions can be written in terms of hypergeometric functions. To achieve this, we start from the Schrödinger equation for the eigenvalue ε^l_κ (6.5)

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\sqrt{\kappa}}{\tan\sqrt{\kappa}r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa}{\sin^2\sqrt{\kappa}r}(m-\beta\,\mathrm{vers}_\kappa r)^2 + 2\varepsilon^l_\kappa\right)R^{\beta,\kappa}_{l,m}(r) = 0. \tag{6.20}$$

If we change to the new variable $x = A\mathrm{vers}_\kappa r$ and factorize the wavefunction as

$$R^{\beta,\kappa}_{l,m}(r) = A^{m/2}\,\mathrm{vers}^{m/2}_\kappa r\,\left(1-\frac{\kappa}{2}\mathrm{vers}_\kappa r\right)^{\beta/\kappa-m/2}\phi^{\beta,\kappa}_{l,m}(r) \qquad \Phi(x) = \phi(r(x)) \tag{6.21}$$

we can rewrite this equation as an hypergeometric-like equation

$$x\left(1-\frac{\kappa}{2A}x\right)\frac{\mathrm{d}^2\Phi^{\beta,\kappa}_{l,m}(x)}{\mathrm{d}x^2} + \left(1+m-\frac{\beta+\kappa}{A}x\right)\frac{\mathrm{d}\Phi^{\beta,\kappa}_{l,m}(x)}{\mathrm{d}x} + \frac{2\varepsilon^l_\kappa-\beta}{2A}\Phi^{\beta,\kappa}_{l,m}(x) = 0. \tag{6.22}$$

Remark that the factor function in (6.21) coincides with the Landau eigenfunctions of the lowest level (6.17). It is also worth noting that equation (6.22) is well behaved when $\kappa \to 0$ giving rise to a confluent hypergeometric equation.

We shall analyse in detail equation (6.22) according to the values of κ:

(i) $\kappa \neq 0$. Choosing $A = \kappa/2$, equation (6.22) turns into the hypergeometric expression

$$x(1-x)\frac{\mathrm{d}^2\Phi^{\beta,\kappa}_{l,m}(x)}{\mathrm{d}x^2} + (1+m-2(\beta/\kappa+1)x)\frac{\mathrm{d}\Phi^{\beta,\kappa}_{l,m}(x)}{\mathrm{d}x} + \frac{2\varepsilon^l_\kappa - \beta}{\kappa}\,\Phi^{\beta,\kappa}_{l,m}(x) = 0 \qquad (6.23)$$

and the solutions we are looking for are given in terms of the hypergeometric function $\Phi^{\beta,\kappa}_{l,m}(x) = F(-l, l+1+2\beta/\kappa, m+1, x)$. However, in order to avoid problems when $1+m \leqslant 0$ we can consider [16]

$$\boldsymbol{F}(-l, l+1+2\beta/\kappa, m+1, x) = \frac{F(-l, l+1+2\beta/\kappa, m+1, x)}{\Gamma(m+1)}$$

which is also solution of (6.23). The complete expression of the local eigenfunctions is

$$\Psi^{\beta,\kappa}_{l,m}(r,\theta) = c_{lm}(\kappa)\frac{\kappa^{m/2}}{2^{m/2}}\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{vers}^{m/2}_\kappa r\left(1-\frac{\kappa}{2}\mathrm{vers}_\kappa r\right)^{\beta/\kappa - m/2}$$
$$\times \boldsymbol{F}\left(-l, l+1+2\beta/\kappa, m+1, \frac{\kappa}{2}\mathrm{vers}_\kappa r\right) \qquad (6.24)$$

where the factor $c_{lm}(\kappa)$ is a normalization constant. With the measure (2.6) its value is

$$c_{lm}(\kappa) = \sqrt{\frac{\kappa\Gamma(l+1+2\beta/\kappa)\Gamma(2l+2+2\beta/\kappa)\Gamma(l+m+1)}{4\pi\Gamma(2l+1+2\beta/\kappa)\Gamma(l-m+1+2\beta/\kappa)}}. \qquad (6.25)$$

Since the above hypergeometric functions can also be expressed in terms of the Jacobi functions, $P_l^{(m,2\beta/\kappa-m)}(\cos\sqrt{\kappa}r)$, we can rewrite (6.24) as

$$\Psi^{\beta,\kappa}_{l,m}(r,\theta) = c_{lm}(\kappa)\frac{\kappa^{m/2}}{2^{m/2}}\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{vers}^{m/2}_\kappa r\left(1-\frac{\kappa}{2}\mathrm{vers}_\kappa r\right)^{\beta/\kappa - m/2}$$
$$\times\frac{l!}{\Gamma(l+m+1)}P_l^{(m,2\beta/\kappa-m)}(\cos\sqrt{\kappa}r). \qquad (6.26)$$

(ii) $\kappa = 0$. Let us assume $\beta \neq 0$ and choose $A = \beta$, then equation (6.22) comes into one of the confluent hypergeometric classes

$$x\,\frac{\mathrm{d}^2\Phi^{\beta,\kappa}_{l,m}(x)}{\mathrm{d}x^2} + (1+m-x)\,\frac{\mathrm{d}\Phi^{\beta,\kappa}_{l,m}(x)}{\mathrm{d}x} + l\,\Phi^{\beta,\kappa}_{l,m}(x) = 0. \qquad (6.27)$$

The appropriate solutions are expressed by means of the confluent hypergeometric function [16] $M(-l, m+1, x)$, or $\boldsymbol{M}(-l, m+1, x) = M(-l, m+1, x)/\Gamma(m+1)$. Consequently, we obtain

$$\Psi^{\beta,0}_{l,m}(r,\theta) = c_{lm}(0)\,\frac{\beta^{m/2}}{2^{m/2}}\,\mathrm{e}^{\mathrm{i}m\theta}\,r^m\mathrm{e}^{-\beta r^2/4}\boldsymbol{M}(-l, m+1, \beta r^2/2) \qquad (6.28)$$

where the normalization constant $c_{lm}(0)$ is given by $c_{lm}(0) = \sqrt{\beta\,\Gamma(l+m+1)/(2\pi\,l!)}$. In terms of Laguerre polynomials, $L_l^m(\beta r^2/2)$, the solutions $\Psi^{\beta,0}_{l,m}(r,\theta)$ can be rewritten as

$$\Psi^{\beta,0}_{l,m}(r,\theta) = c_{lm}(0)\,\frac{\beta^{m/2}}{2^{m/2}}\,\mathrm{e}^{\mathrm{i}m\theta}\,r^m\mathrm{e}^{-\beta r^2/4}\frac{l!}{\Gamma(l+m+1)}L_l^m(\beta r^2/2). \qquad (6.29)$$

It can be checked that the following limits hold when $\kappa \to 0$:

$$\lim_{\kappa\to 0} c_{lm}(\kappa)\frac{\kappa^{m/2}}{2^{m/2}} = c_{lm}(0)\beta^{m/2}$$
$$\lim_{\kappa\to 0}\mathrm{vers}^{m/2}_\kappa r\left(1-\frac{\kappa}{2}\mathrm{vers}_\kappa r\right)^{\beta/\kappa-m/2} = \frac{1}{2^{m/2}}\,r^m\mathrm{e}^{-\beta r^2/4} \qquad (6.30)$$
$$\lim_{\kappa\to 0} F\left(-l, l+1+2\beta/\kappa, m+1, \frac{\kappa}{2}\mathrm{vers}_\kappa r\right) = M(-l, m+1, \beta r^2/2).$$

These limits prove that there exists a well defined contraction process for the local UIR wavefunctions given by

$$\lim_{\kappa\to 0} \Psi^{\beta,\kappa}_{l,m}(r,\theta) = \Psi^{\beta,0}_{l,m}(r,\theta). \tag{6.31}$$

The second family of UIRs admits a similar treatment. Now, factorizing

$$R^{\beta,\kappa}_{l,m}(r) = A^{-m/2}\mathrm{vers}_\kappa^{-m/2} r \left(1 - \frac{\kappa}{2}\mathrm{vers}_\kappa r\right)^{-\beta/\kappa+m/2} \phi^{\beta,\kappa}_{l,m}(r)$$

and performing a variable change $x = A\mathrm{vers}_\kappa r$ we obtain

$$\Psi^{\beta,\kappa}_{l,m}(r,\theta) = c_{l,-m}(\kappa)\frac{2^{m/2}}{\kappa^{m/2}}\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{vers}_\kappa^{-m/2} r\left(1-\frac{\kappa}{2}\mathrm{vers}_\kappa r\right)^{-\beta/\kappa+m/2}$$
$$\times\frac{l!}{\Gamma(l-m+1)}P_l^{(-m,-2\frac{\beta}{\kappa}+m)}(\cos\sqrt{\kappa}r) \tag{6.32}$$

with $c_{l,-m}$ given by (6.25). This function is a solution of equation (6.23) where m and β have been replaced by $-m$ and $-\beta$, respectively.

7. Horocyclic coordinates and variable separation

In this section we shall perform a coordinate separation of the hyperbolic Landau quantum system by means of horocyclic coordinates [18]. The same question, but at the classical level in the complex plane, was addressed in [19, 20]. Nowadays the hyperbolic Landau classical problem continues to be a matter of study from different points of view (see, for instance, [21, 22] and references therein).

There are two points worth mentioning on this subject. The first one is that in the convention of Miller [17] the variable separation in the quantum case is an example of R-separability, i.e. given an equation $E\psi = 0$ the R-separable solutions are, in fact, standard separable solutions of an equivalent equation $E'\phi = 0$ with $E' = R^{-1}ER$ and $\psi = R\phi$. The second remark is that horocyclic coordinates under the contraction $\kappa \to 0$ turn into Cartesian coordinates in the plane.

The horocyclic coordinates $(a, b) \in \mathbb{R}^2$ are associated with the action of the generators J_{01} and $J_{02} + \sqrt{-\kappa}J_{12}$ of $SO_\kappa(3)$ over the point $x_0 = (1, 0, 0)$ of S^2_κ as follows (in this section $\kappa < 0$):

$$(x^0, x^1, x^2)^T = \mathrm{e}^{-\mathrm{i}aJ_{01}}\mathrm{e}^{-\mathrm{i}b(J_{02}+\sqrt{-\kappa}J_{12})}(1, 0, 0)^T \tag{7.1}$$

where the superindex T denotes matrix transposition and the matrix representation of the generators $J_{..}$ is given in expression (2.2). The explicit expression of this coordinate system is

$$\begin{aligned} x^0 &= \cosh\sqrt{-\kappa}a - \kappa\,\frac{b^2}{2}\,\mathrm{e}^{\sqrt{-\kappa}a}\\ x^1 &= \frac{\sinh(\sqrt{-\kappa}a)}{\sqrt{-\kappa}} + \kappa\,\frac{b^2}{2}\,\mathrm{e}^{\sqrt{-\kappa}a}\\ x^2 &= b\,\mathrm{e}^{\sqrt{-\kappa}a}. \end{aligned} \tag{7.2}$$

In the limit $\kappa \to 0$ we recover the Cartesian coordinates of the plane as we mentioned above. These horocyclic coordinates are of 'subgroup type' like the polar geodesic ones used in previous sections. While the former corresponds to the reduction $O(2, 1) \supset T$, where T is the subgroup generated by $J_{02} + \sqrt{-\kappa}J_{12}$, the last one is related to $O(2, 1) \supset O(2)$ (for more details see [23]).

Using horocyclic coordinates the following time-independent Schrödinger equation of the hyperbolic Landau systems holds:

$$\left[\frac{1}{\kappa}\left(\frac{\partial}{\partial a}-\mathrm{i}V_a\right)^2-\frac{1}{\sqrt{-\kappa}}\left(\frac{\partial}{\partial a}-\mathrm{i}V_a\right)+\frac{\mathrm{e}^{-2\sqrt{-\kappa}a}}{\kappa}\left(\frac{\partial}{\partial b}-\mathrm{i}V_b\right)^2\right]\Phi(a,b)=-\frac{E}{\kappa}\Phi(a,b) \tag{7.3}$$

where

$$V_a=\frac{2\kappa\beta b}{2-\kappa b^2\mathrm{e}^{\sqrt{-\kappa}a}+2\cosh\sqrt{-\kappa}a}\qquad V_b=\frac{\beta(-1+\mathrm{e}^{-2\sqrt{\kappa}a}(1-\kappa b^2))}{2-\kappa b^2\mathrm{e}^{\sqrt{-\kappa}a}+2\cosh\sqrt{-\kappa}a} \tag{7.4}$$

are the electromagnetic potential components in these coordinates.

Taking under consideration the fact that the wavefunction has the form

$$\Phi(a,b)=\exp\left[\frac{\beta}{\sqrt{-\kappa}}\left(\sqrt{-\kappa}\,b-2\arctan\frac{\sqrt{-\kappa}b\mathrm{e}^{\sqrt{-\kappa}a}}{1+\mathrm{e}^{\sqrt{-\kappa}a}}\right)\right]\psi(a)\phi(b) \tag{7.5}$$

and

$$\phi(b)=\mathrm{e}^{\mathrm{i}\lambda b}\qquad \lambda\in\mathbb{R} \tag{7.6}$$

we obtain, after rescaling multiplying by κ, a new differential equation only in the variable a, i.e. coordinates a and b allow a variable separation with λ as the separation constant

$$\left[-\frac{\partial^2}{\partial a^2}-\sqrt{-\kappa}\frac{\partial}{\partial a}-\frac{1}{\kappa}\mathrm{e}^{-2\sqrt{-\kappa}a}(\beta(\mathrm{e}^{\sqrt{-\kappa}a}-1)-\sqrt{-\kappa}\lambda)^2-E\right]\psi(a)=0. \tag{7.7}$$

Note that the differential operator $\overline{J}_{02}+\sqrt{-\kappa}\;\overline{J}_{12}$ (see expression (3.2)) is straightened out to the form $-\mathrm{i}\partial_b+f(a,b)$, where

$$f(a,b)=\frac{2\sinh(\sqrt{-\kappa}a)+\kappa b^2\mathrm{e}^{\sqrt{-\kappa}a}}{\sqrt{-\kappa}(2-\kappa\mathrm{e}^{\sqrt{-\kappa}a}+2\cosh(\sqrt{-\kappa}a))}$$

corresponds to the term $W_{02}(x)$ of $\overline{J}_{02}$ in (3.2). In order to eliminate this function we have introduced the phase (7.5) of $\Phi(a,b)$, which performs the R-separation and it is defined by $\exp(\int f(a,b)\,\mathrm{d}b)$. The operator $\overline{J}_{01}$ given by (3.2) becomes $\overline{J}_{01}=-\mathrm{i}\partial_a+\beta b$.

In the limit $\kappa\to 0$ of equation (7.7) we recover the harmonic oscillator Schrödinger equation of unit mass, frequency $\omega=|\beta|$, energy $E/2$ and origin $a=\lambda/\beta$.

Equation (7.7) can be set into the standard expression

$$\left[-\frac{\partial^2}{\partial a^2}-\left(E'-\frac{\beta^2}{\kappa}(-\mathrm{e}^{-2\sqrt{-\kappa}a}+2\mathrm{e}^{-\sqrt{-\kappa}a})\right)\right]\psi(a)=0 \tag{7.8}$$

with

$$E'=E-\frac{\beta^2}{\sqrt{-\kappa}}+\frac{\kappa}{4} \tag{7.9}$$

by means of the following transformations:

- coordinate translation $a\to a-\alpha/\sqrt{-\kappa}$, with $\mathrm{e}^{\alpha}=(\beta+\sqrt{-\kappa}\lambda)^2$ and $\mathrm{sign}(\beta)=\mathrm{sign}(\lambda)$,
- $\psi(a)\to\mathrm{e}^{-\sqrt{-\kappa}/2}\psi(a)$,
- new coordinate translation $a\to a-\gamma/\sqrt{-\kappa}$, where $|\beta|=\mathrm{e}^{\gamma}$.

Equation (7.8) corresponds to the Schrödinger equation of a particle of unit mass moving in a Morse potential $\beta^2/2\kappa(-\mathrm{e}^{-2\sqrt{-\kappa}a} + 2\mathrm{e}^{-\sqrt{-\kappa}a})$ [1].

As it is well known, the energy spectrum of the Morse potential has two parts: one discrete ($E' < 0$) and the other continuous ($E' \geqslant 0$). Hence, we have from (7.9) that the energy for the continuous spectrum of the Landau problem corresponds to

$$E \geqslant \frac{\beta^2}{\sqrt{-\kappa}} - \frac{\kappa}{4}. \tag{7.10}$$

To obtain the discrete spectrum we proceed as follows. The variable change $\xi = -(2\sqrt{\beta^2}/\kappa)\mathrm{e}^{-\sqrt{-\kappa}a}$ and the factorization $\psi(a) = \mathrm{e}^{-\xi/2}\xi^s f(\xi)$ give the equation

$$\xi f''(\xi) + (2s + 1 - \xi) f'(\xi) + l f(\xi) = 0 \tag{7.11}$$

where $s = \sqrt{E'/\kappa}$ and $l = -\sqrt{\beta^2}/\kappa - (s + 1/2)$. The confluent hypergeometric function $f(\xi) = M(-l, 2s + 1, \xi)$ with $l \in \mathbb{Z}^{\geqslant 0}$ is a suitable solution of equation (7.11). The energy spectum is

$$\mathcal{E}_l = \frac{E_l}{2} = |\beta| \left(l + \frac{1}{2} \right) + \frac{\kappa}{2} l(l + 1) \tag{7.12}$$

that agrees with expressions (6.5) and (6.10). The number of Landau levels is finite since

$$0 \leqslant l < \frac{|\beta|}{|\kappa|} - \frac{1}{2} \tag{7.13}$$

as expected from the discussion presented in section 6.2.

8. Algebraic analysis of Landau equations

8.1. Ladder operators for energy levels

For the planar Landau systems, besides the (shift) operators that act inside each energy level changing only the values of m, there are also other types of (ladder) operators connecting states of different energies. We shall show in the following that the Landau systems with $\kappa \neq 0$ (thus, including the spherical and hyperbolic systems) also admit ladder operators such that in the limit $\kappa \to 0$ come into those associated with the planar case.

Let us multiply the eigenvalue equation (6.20) by the function $(\sin^2 \sqrt{\kappa} r)/\kappa$. The resulting differential equation

$$\begin{aligned} \mathcal{E}_l\, R^{\beta,\kappa}_{l,m}(r) \equiv \Bigg(&\frac{\sin^2 \sqrt{\kappa} r}{\kappa} \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\sin \sqrt{\kappa} r\, \cos \sqrt{\kappa} r}{\sqrt{\kappa}} \frac{\mathrm{d}}{\mathrm{d}r} - (m - \beta\, \mathrm{vers}_\kappa r)^2 \\ &+ \frac{\varepsilon^l_\kappa \sin^2 \sqrt{\kappa} r}{\kappa} \Bigg) R^{\beta,\kappa}_{l,m}(r) = 0 \end{aligned} \tag{8.1}$$

can be factorized as follows:

$$\left\{ \left(\frac{\sin \sqrt{\kappa} r}{\sqrt{\kappa}} \frac{\mathrm{d}}{\mathrm{d}r} + \mu_l \cos \sqrt{\kappa} r + \nu_l \right) \left(\frac{\sin \sqrt{\kappa} r}{\sqrt{\kappa}} \frac{\mathrm{d}}{\mathrm{d}r} - \mu_l \cos \sqrt{\kappa} r - \nu_l \right) + \delta_l \right\} R^{\beta,\kappa}_{l,m}(r) = 0 \tag{8.2}$$

with

$$\mu_l = \frac{\beta}{\kappa} + l \qquad \nu_l = -\frac{\beta}{\kappa} + \frac{\beta(m + l)}{\beta + \kappa l} \qquad \delta_l = \frac{2\beta l(m + l)}{\beta + \kappa l} + \frac{\beta^2(m + l)^2}{(\beta + \kappa l)^2} - m^2 + l^2. \tag{8.3}$$

So, the factorization of the second-order differential operator $\mathcal{E}_l$ (8.1) can also be written schematically in the form

$$\mathcal{E}_l = A_l^+ A_l^- + \delta_l \tag{8.4}$$

where the label l corresponds to the energy value ε_κ^l in equation (6.5) keeping fixed κ and m. It can be checked (for instance, through the symmetry change $l \to -l - 2\beta/\kappa - 1$), according to the previous notation, that

$$\mathcal{E}_{l-1} = A_l^- A_l^+ + \delta_l = A_{l-1}^+ A_{l-1}^- + \delta_{l-1}. \tag{8.5}$$

This means that the operator A_l^- connects the eigenfunction space of eigenvalue ε_κ^l to the one corresponding to ε_κ^{l-1}, while A_l^+ acts in the opposite direction. In fact, when $A_l^\pm$ do not spoil the normalization conditions, they will link the radial eigenfunctions $R_{l,m}^{\beta,\kappa}(r)$ and $R_{l-1,m}^{\beta,\kappa}(r)$, up to a factor. By means of the operator set $\{A_l^\pm\}$ we can define free-index operators $A^\pm$ [17] with commutation rules

$$[A^-, A^+] = \Delta(L) \tag{8.6}$$

where the involved operators in (8.6) when acting on $R_{l,m}^{\beta,\kappa}(r)$ must be read in the form

$$[A^-, A^+] = A_{l+1}^- A_{l+1}^+ - A_l^+ A_l^- \qquad \Delta(L) = \delta_l - \delta_{l+1} \tag{8.7}$$

where L is a diagonal operator, $L\, R_{l,m}^{\beta,\kappa} = l\, R_{l,m}^{\beta,\kappa}$.

In the limit $\kappa \to 0$ all the elements in (8.2), (8.3) are well defined and we recover the planar Landau ladder operators:

$$A_l^\pm \to r\frac{\mathrm{d}}{\mathrm{d}r} \mp \frac{\beta}{2} r^2 \pm (2l + m) \qquad \delta_l \to 4l^2 + 4lm. \tag{8.8}$$

Now, the so-obtained free-index operators $\{A^+, A^-, L\}$, for $\kappa = 0$, close a Lie algebra isomorphic to $so(2, 1)$. However, when $\kappa \neq 0$, as can be seen from (8.4) and (8.6), these operators generate an associative algebra but not a Lie algebra.

There is a freedom in normalizing the operators $\{A^+, A^-\}$ of (8.7), so that if we change to the set $\{\tilde{A}_l^+ = \sqrt{\beta + \kappa l} A_l^+, \tilde{A}_l^- = A_l^- \sqrt{\beta + \kappa l}\}$, the new pair $\{\tilde{A}^+, \tilde{A}^-\}$ will now satisfy cubic commutation relations. This is the kind of algebra related to the isotropic oscillator in curved spaces discussed in references [24–26]. Such a connection seems very suggestive since as it is known the Landau system in the plane is closely related to the 2D oscillator.

8.2. Annihilation lines and solution sectors

We shall study some consequences of the factorization (8.2) that can help in computing the eigenfunctions of the Landau wave equation by a new procedure. This section can be seen as an application of the refined factorization method [27, 28].

The main role of our discussion is played by the expression of δ_l in (8.3). Although apparently complicated, it is responsible for the spectrum 'shape' of the Landau systems in a way that will be precised below. Let us consider the solutions in l of the equation

$$\delta_l = \frac{2\beta l(m + l)}{\beta + \kappa l} + \frac{\beta^2(m + l)^2}{(\beta + \kappa l)^2} - m^2 + l^2 = 0. \tag{8.9}$$

For these values, according to (8.2), the states ψ_l^- annihilated by A_l^-, $A_l^- \psi_l^- = 0$, satisfy the equation (8.1) $\mathcal{E}_l \psi_l^- = 0$, in other words, they are solutions of the Landau eigenequation. In the same way the reasoning follows through for the solutions of $\delta_{l+1} = 0$. In this case the states ψ_l^+ such that $A_{l+1}^+ \psi_l^+ = 0$ will be also solutions of (8.1).

Based on these considerations, once fixed κ, the solutions of $\delta_l = 0$ ($\delta_{l+1} = 0$) in the plane (m, l) will be called annihilation lines of A^- (A^+). These lines provide immediate solutions of the Landau systems, but it is still necessary to specify carefully which ones are normalizable. In this case such states will constitute vacuum states that can be used to build the whole spectrum by applying ladder operators. It is also important to determine when the action of such operators will lead us out of the normalizable sector.

The solutions to the equation $\delta_l = 0$ are straight lines. For each of these lines we can build the operators A_l^- according to (8.2)–(8.4) and find the cases where the states ψ_l^- are normalizable. The results are summarized below depending on the κ values (we have always assumed that $\beta > 0$):

- ($\kappa = 0$)

$$\begin{array}{lll} & \text{solutions} & \psi_l^- \text{ normalizable if} \\ \text{(i)} & l = -m & m \leqslant 0 \\ \text{(ii)} & l = 0 & m \geqslant 0 \end{array} \tag{8.10}$$

- ($\kappa > 0$)

$$\begin{array}{lll} & \text{solutions} & \psi_l^- \text{ normalizable if} \\ \text{(i)} & l = -m & m \leqslant 0 \\ \text{(ii)} & l = 0 & 0 \leqslant m \leqslant 2\beta/\kappa \\ \text{(iii)} & l = -2\beta/\kappa & \text{never} \\ \text{(iv)} & l = m - 2\beta/\kappa & m \geqslant 2\beta/\kappa. \end{array} \tag{8.11}$$

  Since the UIRs of $SO_\kappa(3)$ ($\kappa > 0$), computed in section 6.1, restrict the parameter values to $2\beta/\kappa \in \mathbb{Z}^+, l \in \mathbb{Z}^+, -l \leqslant m \leqslant 2\beta/\kappa$, we see that they coincide with those displayed in the above display. In fact, we can check that the normalizable eigenfunctions ψ_l^- defined on the annihilation lines are the same than those previously found in sections 6.2 and 6.3. Therefore, the shift and ladder operators are consistent in the sense that they act in the same space of physical states.

- ($\kappa < 0$)

$$\begin{array}{lll} & \text{solutions} & \psi_l^- \text{ normalizable if} \\ \text{(i)} & l = -m & \beta/\kappa + 1/2 < m \leqslant 0 \\ \text{(ii)} & l = 0 & 0 \leqslant m \\ \text{(iii)} & l = -2\beta/\kappa & \text{never} \\ \text{(iv)} & l = m - 2\beta/\kappa & \text{never.} \end{array} \tag{8.12}$$

  The same comments can be made with respect to this display: all the restrictions and wavefunctions are consistent with the unitary representations of $SO_\kappa(3)$ with $\kappa < 0$.

The sector of physical eigenstates bounded by these lines are depicted in figure 1. The parameters associated with the normalizable wavefunctions that constitute a lattice inside such sectors are shown schematically in figure 2.

In order to look for the annihilation lines of the operator A^+ one can use the symmetry $l \to -l - 2\beta/\kappa - 1$ to get

$$\begin{array}{ll} \text{(i}'\text{)}\ l = m - 2\beta/\kappa - 1 & \text{(ii}'\text{)}\ l = -2\beta/\kappa - 1, \\ \text{(iii}'\text{)}\ l = -1 & \text{(iv}'\text{)}\ l = -m - 1. \end{array} \tag{8.13}$$

These lines can be used to give an equivalent description for the unitary representations corresponding to highest weight $l \in \mathbb{Z}^-$ quoted in section 6.1, so we will not refer to them any longer. The graphs of physical sectors, lines and states are symmetric, with respect to the l axis, to the cases derived from A^-.

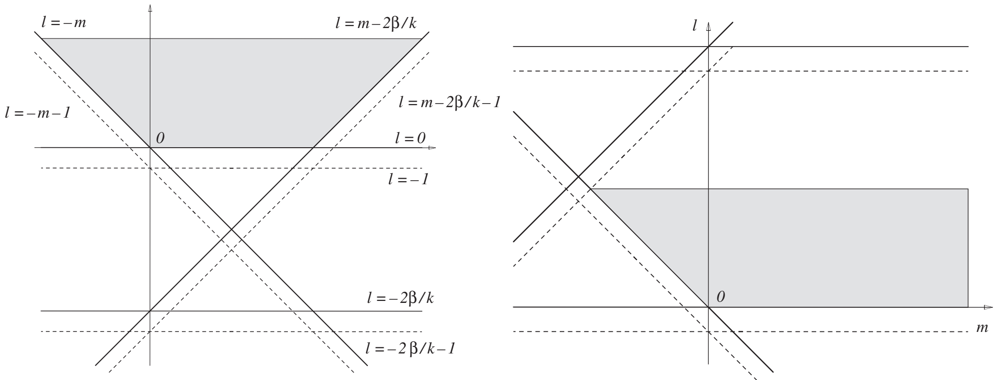

Figure 1. Physical sectors in grey for $\kappa > 0$ (left), and $\kappa < 0$ (right), together with annihilation lines for A^- (solid lines) and A^+ (dashed lines).

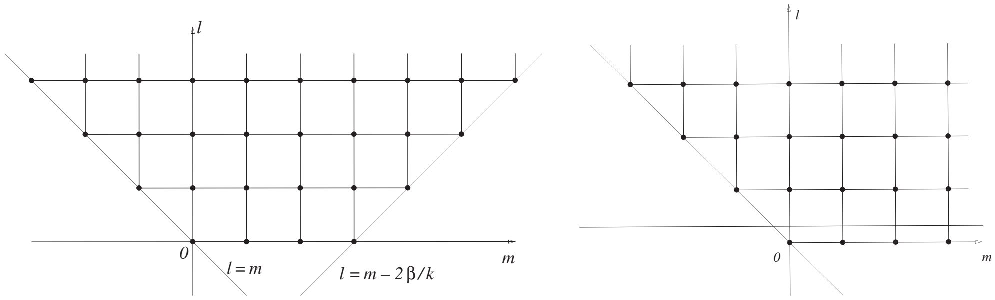

Figure 2. Lattice of normalizable states for $\kappa > 0$ (left), and $\kappa < 0$ (right).

8.3. Moving states

If we exclude the north pole of S^2_κ (or both, if $\kappa > 0$), then the set of local realizations of the Lie algebra $so_\kappa(3)$ acting on differentiable functions is bigger, since it is parametrized by two real labels b and λ, as it is shown in appendix B.

Let us concentrate on the case $\kappa \neq 0$. The θ-component of the invariant gauge potential is $A_\theta = b/\kappa - \lambda\cos(\sqrt{\kappa}r)$, hence changes in the parameter b lead to the same field (see appendix C and section 5.1). Moreover, if we keep the notation of previous sections $\lambda = \beta/\kappa$, now the potentials can be rewritten as $A_\theta = \beta/\kappa\ \mathrm{vers}_\kappa(r) + \rho$ with $\rho = b/\kappa - \lambda + n$. Also, we can perform a gauge transformation in the class of differentiable functions changing b/κ into $b/\kappa + n$, $n \in \mathbb{Z}$. Therefore, the classes of gauge equivalent potentials can be characterized by a real parameter $0 \leqslant \rho < 1$ [29].

We can change the point of view and leave one potential fixed for each field (choosing for instance $\rho = 0$) but defining different classes of carrier spaces characterized by the wavefunctions satisfying the boundary condition

$$\Psi(r, \theta + 2\pi) = \mathrm{e}^{\mathrm{i}2\pi\alpha}\Psi(r, \theta) \qquad 0 \leqslant \alpha < 1. \tag{8.14}$$

So, we have transferred the parameter ρ, labelling the classes of gauge potentials, to a phase $\mathrm{e}^{\mathrm{i}2\pi\alpha}$ of the wavefunctions. For any α we have a differential realization of the Lie algebra $so_\kappa(3)$ with an invariant gauge potential, but no longer a realization of the group $SO_\kappa(3)$. Nevertheless, we obtain a solvable system, whose eigenfunctions are the so-called 'moving states', which have interest in the interpretation of the Hall effect [5]. Now, the equation for

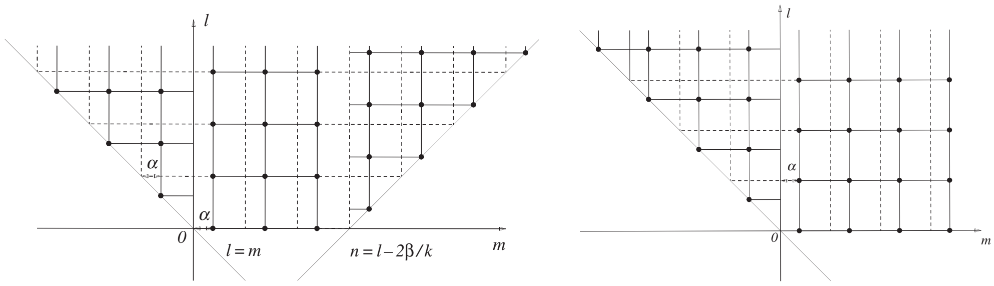

Figure 3. Lattice of normalizable states with aperiodic boundary conditions ($\alpha \neq 0$) for $\kappa > 0$ (left), and $\kappa < 0$ (right).

the spectrum keeps the same form as (5.5) or equivalently (8.1) but where the parameter m must be substituted by $m + \alpha$.

In this case all the considerations about the ladder operator method also remain valid with the same annihilation lines except that the physical vacuums on these lines are parametrized by $m = n + \alpha$, $n \in \mathbb{Z}$, keeping the restrictions (8.10)–(8.12). However, there are some properties that have changed drastically. When $\alpha = 0$ each physical sector is invariant under the action of the operators $\{J^{\pm}, A^{\pm}\}$ (for $\kappa < 0$, in a two-fold way), and each state is connected with any other by means of such operators as we see in the above section. But when $\alpha \neq 0$ each physical sector is broken into subsectors with the following modifications: (i) the states of each subsector are linked by means of $\{J^{\pm}, A^{\pm}\}$, but states belonging to different subsectors are not connected any more. (ii) Each subsector is not invariant under all the shift and ladder operators, so that the action of some of them may lead to non-physical states. These properties are illustrated separately in figure 3 for $\kappa > 0$ and $\kappa < 0$. Note that all these features can be obtained, of course, in the frame of the hypergeometric equation given in section 6.4, but we would loose track of the shift and ladder operators that are so useful in describing the change in the spectrum.

From figure 3 one can easily understand the 'index' [5] associated with each energy level. This index is defined as the difference between the number of states that join and leave an energy level when the parameter α increases by a period. For $\kappa \leqslant 0$ the index is 1, but in the compact case ($\kappa > 0$) the index is 0.

9. Remarks and conclusions

In order to physically interpret our results, we will consider dimensions for the Hamiltonian $H_\kappa = \overline{C}/2$ with the Casimir given in (2.8). By means of a multiplicative factor and the identification

$$\beta = \tau q \frac{|\boldsymbol{B}|}{\hbar c} \tag{9.1}$$

where q is the charge of the physical system, $|\boldsymbol{B}|$ is the intensity of the magnetic field and $\tau = +1$ or -1 indicates that the magnetic field points in or out the direction of J_{12}, we can rewrite H_κ in the form

$$H_\kappa = \frac{\hbar^2}{2m_0}(J_{01}^2 + J_{02}^2 + \kappa J_{12}^2) - \tau \frac{\hbar q}{m_0} |\boldsymbol{B}| J_{12}. \tag{9.2}$$

The spectrum of the Hamiltonian (9.2) is

$$E_l^\kappa = \frac{|q|\hbar}{m_0 c}|\boldsymbol{B}|\left(l + \frac{1}{2}\right) + \frac{\hbar^2\kappa}{2m_0}l(l+1). \tag{9.3}$$

Note that we have put together expressions (6.5) and (6.10). The second term of the energy (9.3) has a marked geometric meaning since it depends on κ which is related to the curvature radius, R, of the configuration space ($|\kappa| = 1/R^2$). In the limit $\kappa \to 0$ this term disappears as is the case in the Euclidean plane.

When $\kappa \neq 0$ we can consider a monopole with an associated radial magnetic field $\boldsymbol{B}$. The Dirac quantization condition [14] gives

$$|\boldsymbol{B}| = \frac{\hbar n}{|e|R^2} = \frac{\hbar n|\kappa|}{|e|} \tag{9.4}$$

with n denotes a natural number and e the elementary negative charge. Note that if we require that in the limit $\kappa \to 0$ the field $\boldsymbol{B}$ remains finite, $n \to \infty$ in order to keep the term κn constant.

Now the spectrum is

$$E_l^{n,\eta} = \frac{|q|\hbar}{m_0 c}|\boldsymbol{B}|\left(l + \frac{1}{2}\right) + \eta\frac{\hbar|e||\boldsymbol{B}|}{2m_0 cn}l(l+1) \tag{9.5}$$

where $\eta = +1$ or -1 according to the configuration space has positive or negative curvature, respectively.

The Landau systems considered in this paper can be obtained from the relevant symmetry groups of the involved magnetic fields by means of their local realizations. Each irreducible component inside a class of local realizations has labels (β, l, κ) whose meaning is the following: (i) a real parameter β proportional to the intensity of an external magnetic field interacting with the quantum system (this intensity is quantized for $\kappa > 0$, which implies the quantization of the magnetic charge); (ii) a positive integer l that determines the energy of the Landau level and characterizes the bounded (discrete) representations of the magnetic groups; and (iii) a real label κ, measuring the curvature of the 2D configuration space, whose standard values $\kappa = 1, 0, -1$ correspond to the three Landau systems, spherical, planar and hyperbolic, respectively. The expressions in terms of κ facilitates the comparison among these systems showing their analogies and differences as well. Besides this, such expressions have sense for any real value of κ. This property tells us how to correctly connect the spherical and/or hyperbolic systems to the well known planar Landau system by the contraction procedure $\kappa \to 0$. Note, as we mentioned before, that κ also shows how the geometry of the configuration space contributes to the energy eigenvalues by means of the term $\kappa l(l+1)$.

To summarize, we have attached a physical meaning to all the parameters labelling each class of local realizations up to gauge equivalence. Let us insist here on the role played by the local character of this classification. Once fixed κ, there are several choices for the remaining parameters, β and l, giving rise to (globally) equivalent irreducible representations. For instance, if $\kappa > 0$, as long as $l + \beta/\kappa = j$, with j being a fixed positive half-integer we will always get $(2j+1)$-D equivalent irreducible representations of $SU(2)$, the universal covering of $SO(3)$. However, different elections of the pairs (β, l) fulfilling such a condition do not belong to the same class of *local equivalence*. This is the reason why they should be considered as describing non-equivalent physical systems, e.g., systems with different energy (l) evolving under a different magnetic field (β). In mathematical terms we would say that, for $\kappa > 0$, the hypergeometric functions include in *several ways* each representation of $SU(2)$. To see the meaning of this point more explicitly take, for instance, $\kappa = 1$, and make the following two choices: (1) ($l = j, \beta = 0$); and (2) ($l = 0, \beta = j$). In the first case the Landau energy level is $E_{(1)} = (j+1)j/2$, while in the second one $E_{(2)} = j/2$, so both systems are not equivalent

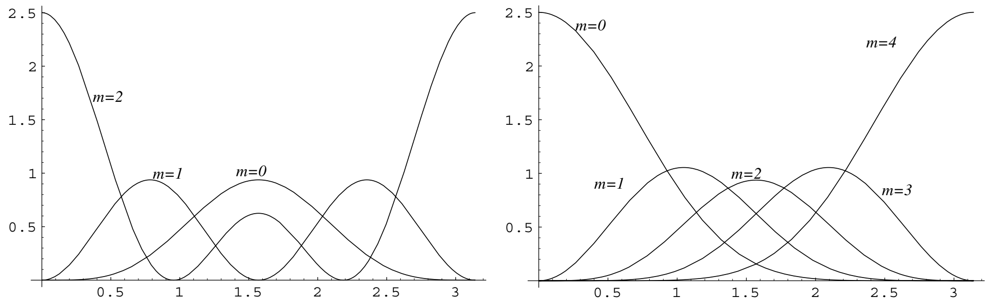

Figure 4. Density probability $|R^{\beta,\kappa=1}_{l,m}(r)|^2$ of the eigenfunctions $m = 0, \pm 1, \pm 2$ ($|R^{0,\kappa=1}_{l,m}(r)|^2 = |R^{0,\kappa=1}_{l,-m}(r)|^2$) in the representation $l = 2, \beta = 0$ (left), and those with $m = 0, \ldots, 4$ for $l = 0, \beta = 2$ (right); $r \in [0, \pi]$.

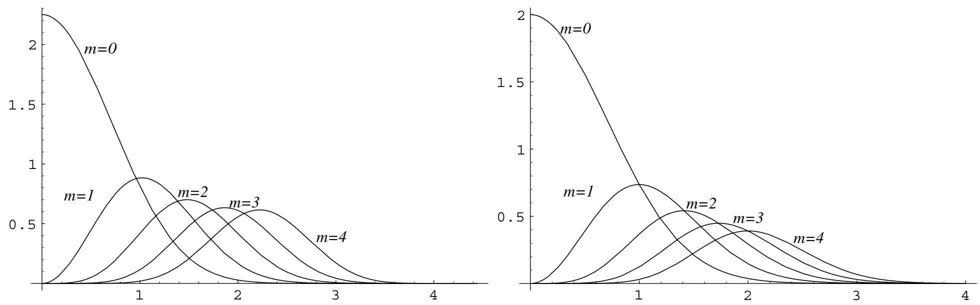

Figure 5. Density probability $|R^{2,\kappa}_{0,m}(r)|^2$ of the eigenfunctions $m = 0, \ldots, 4$ of the representations $l = 0, \beta = 2$ for $\kappa = 1/2$, $r \in [0, \sqrt{2}\pi]$ (left), and for $\kappa = 0$, $r \in [0, +\infty)$ (right).

from a physical point of view. Figure 4 displays the density probability corresponding to each of the five eigenfunctions for these two cases when $j = 2$. This is another way to make explicit the local inequivalence of both sets of eigenfunctions.

It is also interesting to see how the initial eigenfunctions starting from the sphere $S_{\kappa=1} \equiv S^2$ evolve into those defined in the plane $S_{\kappa=0} \equiv E^2$. We illustrate this behaviour for the radial density of the wavefunctions characterized by $l = 0$, $m = 0, \ldots, 4$ and the values $\kappa = 1$, $r \in [0, \pi]$ in figure 4 (right), $\kappa = 1/2$, $r \in [0, \sqrt{2}\pi]$ in figure 5 (left), and $\kappa = 0$, $r \in [0, +\infty)$ in figure 5 (right).

The continuous spectrum of the hyperbolic Landau quantum systems can be easily understood through horocyclic coordinates that allow one to transform these hyperbolic systems into Morse systems. These coordinates display R-separability, however they contract to Cartesian coordinates on the plane.

From the symmetry group we have derived operators, that leave invariant each eigenspace, but we also have obtained ladder operators $\{A^{\pm}\}$ (not related to the space symmetry) linking consecutive eigenspaces $\varepsilon^l_\kappa \to \varepsilon^{l\pm1}_\kappa$. These new operators act in the same lattice of physical states defined by the group generators $\{J^{\pm}\}$. The main interest of the ladder operators is that they show the link of Landau systems with oscillators on curved spaces and allow one to understand in a simple way the spectrum for quasi-periodic wavefunctions or moving states.

Appendix A. Local realizations of symmetry groups

In QM the elements g of the symmetry group G of a quantum system are represented by local unitary operators $U(g)$ acting on the space of wavefunctions ψ defined on the space–time manifold (a homogeneous space of G) X in the form

$$\psi'(x') \equiv (U(g)\psi)(g\,x) = A(g,x)\psi(x) \qquad g \in G,\ x \in X \tag{A.1}$$

where $A(g,x)$ is a matrix-valued function. In the particular case of one-component wavefunctions $A(g,x)$ is simply a phase function, i.e. $A(g,x) = \mathrm{e}^{\mathrm{i}\zeta(g,x)}$ (in the following we will only consider one-component wavefunctions).

The operators $U(g)$ (A.1) close, in general, not a true representation but a projective (or 'up to a factor') representation of G [13] that, henceforth, we shall call *local realization*,

$$U(g_2)U(g_1) = \omega(g_2,g_1)U(g_2g_1) \qquad g_2, g_1 \in G. \tag{A.2}$$

The function $\omega : G \times G \to U(1)$ is the factor system of the realization and it is a 2-cocycle, i.e. $\omega \in \boldsymbol{Z}^2(G,U(1))$. The notation $\omega(g_2,g_1) = \exp\{\mathrm{i}\,\xi(g_2,g_1)\} \in U(1)$ is often used, where $\xi(g_2,g_1) \in \mathbb{R}$ is called the exponent of ω. Only if $\omega(g_2,g_1) = 1, \forall g_2, g_1 \in G$, the realization U is, in fact, a true representation.

The equivalence of local realizations must keep the local character and it is called *gauge, or local, equivalence*. Given two local realizations U and U' of G, they are said to be gauge equivalent if there is a function $\lambda : G \to U(1)$ and a linear operator T acting locally in the carrier space, i.e. $[Tf](x) = T(x)f(x)$, with $T(x)$ a phase factor, such that

$$U'(g) = \lambda(g)TU(g)T^{-1} \qquad \forall g \in G. \tag{A.3}$$

Their corresponding phase functions are related by

$$\mathrm{e}^{\mathrm{i}\zeta'(g,x)} = \lambda(g)T(gx)\mathrm{e}^{\mathrm{i}\zeta(g,x)}T^{-1}(x). \tag{A.4}$$

The factor systems ω and ω' associated with two equivalent realizations, U and U' of G, are said to be equivalent; they satisfy

$$\omega'(g_2,g_1) = \lambda^{-1}(g_2,g_1)\lambda(g_2)\lambda(g_1)\omega(g_2,g_1) \qquad \forall g_2, g_1 \in G. \tag{A.5}$$

In particular, a factor system is trivial if it is equivalent to 1; in other words, it is a 2-coboundary, $\omega \in \boldsymbol{B}^2(G,U(1))$. The quotient $\boldsymbol{H}^2(G,U(1)) = \boldsymbol{Z}^2(G,U(1))/\boldsymbol{B}^2(G,U(1))$ is the second cohomology group of G and it takes part in the characterization of the equivalence classes of the unitary irreducible projective representations of G.

The classification of all the local realizations up to gauge equivalence has been solved in general terms (see [30] and references therein). As a first step the local realizations are linearized, e.g., instead of computing directly the representations up to a factor of G we can get them from the linear local representations of a new group $\overline{G}$. The local representations $\overline{U}$ of $\overline{G}$ originate the local realizations U of G once a section $s : G \to \overline{G}$, has been chosen and provided that $\overline{U}|_{\hat{\boldsymbol{H}}^2(G,U(1))} \subset U(1)$. Then, the realization U associated with the representation $\overline{U}$ is given by

$$U(g) := \overline{U}(s(g)). \tag{A.6}$$

The group $\overline{G}$ is a central extension of G by an Abelian group A. It can be shown that A is the dual of (a subgroup of) the second cohomology group of G, $\hat{\boldsymbol{H}}^2(G,U(1))$. Therefore, a necessary ingredient is $\boldsymbol{H}^2(G,U(1))$ whose computation can be performed by solving the equivalent problem of the central extensions of G by $U(1)$ [31]. If G is simple (such as it is $SO(3)$) the representation group $\overline{G}$ is simply the universal covering group ($SU(2)$ in this case).

Appendix B. Local representations of the magnetic Landau groups

In our particular case, $SO_\kappa(3)$ has only a nontrivial central extension when $\kappa = 0$ described by the commutator $[J_{01}, J_{02}] = \lambda I$, the other two commutators are nonzero and the possible extension parameters λ_{ij} related to them can be reabsorbed by an equivalence. In conclusion, $\boldsymbol{H}^2(so_{\kappa=0}(3), \mathbb{R}) = \mathbb{R}$, and $\boldsymbol{H}^2(so_\kappa(3), \mathbb{R}) = 0$ if $\kappa \neq 0$. In any case, even when $\kappa \neq 0$, we shall take into account the (trivial) extension $[J_{01}, J_{02}] = \mathrm{i}\kappa J_{12} + \lambda I$, in order to have a common formalism for any κ-value. The group $\overline{SO}_\kappa(3)$ used to build the realizations of $SO_\kappa(3)$ according to appendix A will be referred to as the 'magnetic Landau group'. Its Lie algebra, $\overline{so}_\kappa(3)$, is an extension of $so_\kappa(3)$ by $\mathbb{R}$, with Lie commutators (now the central element is called B)

$$\begin{aligned} &[\overline{J}_{01}, \overline{J}_{02}] = \mathrm{i}\kappa\,\overline{J}_{12} + \mathrm{i}B, \qquad &&[\overline{J}_{12}, \overline{J}_{01}] = \mathrm{i}\overline{J}_{02}, \\ &[\overline{J}_{12}, \overline{J}_{02}] = -\overline{J}_{01}, &&[., B] = 0. \end{aligned} \tag{B.1}$$

Now we are in the position to characterize the classes of local realizations of $SO_\kappa(3)$ up to gauge equivalence in terms of the local representations of $\overline{SO}_\kappa(3)$.

Theorem. *The local realizations, $U_{\lambda,\beta}$, of $SO_\kappa(3)$ are obtained by means of the representations of $\overline{SO}_\kappa(3)$ induced from the 1D representations*

$$D_{\lambda,b}(\phi, \zeta) = \mathrm{e}^{-\mathrm{i}(\lambda\phi + b\zeta)} \qquad b, \lambda \in \mathbb{R} \tag{B.2}$$

of the Abelian isotropy subgroup of x_0, generated by $\overline{J}_{12}$ and B.

The induced representations can be straightforwardly computed (see [7] where the local realizations for the Euclidean group are worked out). So, we supply below the representations of the $\overline{so}_\kappa(3)$ generators (3.1), with $\kappa \neq 0$, obtained by induction from (B.2):

$$\begin{aligned} J_{01} &= J_{01}(r, \theta) - \left(\lambda - \frac{b}{\kappa}\cos\sqrt{\kappa}r\right)\frac{\sqrt{\kappa}\sin\theta}{\sin\sqrt{\kappa}\,r} \\ J_{02} &= J_{02}(r, \theta) + \left(\lambda - \frac{b}{\kappa}\cos\sqrt{\kappa}r\right)\frac{\sqrt{\kappa}\cos\theta}{\sin\sqrt{\kappa}\,r} \\ J_{12} &= J_{12}(r, \theta), \qquad B = -b. \end{aligned} \tag{B.3}$$

Here, some remarks concerning expressions (B.3) are appropriate:

(i) $\kappa \neq 0$. The label λ in the representation (B.2) of $\overline{SO}(2)$ must be a half-integer for $\kappa > 0$ (the universal covering of $SO(3)$ has centre $\mathbb{Z}_2$) or real for $\kappa < 0$ (the corresponding universal covering has centre $\mathbb{Z}$). We will take a half-integer value to include both cases. The value of λ determines a class of local equivalence, while b is an irrelevant real parameter that can be gauged away. However, we can choose b in an appropriate way: if we take $\lambda = b/\kappa \equiv \beta/\kappa$, the generators (B.3) become differentiable in the north pole. A second reason for this choice is given below.

(ii) $\kappa = 0$. In this case λ is irrelevant (it can be arbitrarily changed by means of a pseudoequivalence (A.3)), but the parameter $b \in \mathbb{R}$ now becomes significant and determines the class of local realization. If we look at the realization (B.3) of the generators for $\kappa \neq 0$ we see that it does not have a well defined limit when $\kappa \to 0$. If we want to obtain the correct nontrivial expressions for $\kappa = 0$ in this way we must choose $\lambda = b/\kappa \equiv \beta/\kappa$, as stated above. With this choice we have only one parameter, β, that determines the local class for any κ. This is the final result shown in (3.2).

Appendix C. Fibre bundles and local representations

Let us consider the principal bundle $\overline{SO}_\kappa(3)(S^2_\kappa, \pi, SO(2)\otimes\mathbb{R})$, with total space $\overline{SO}_\kappa(3)$, base space S^2_κ, projection $\pi : \overline{SO}_\kappa(3) \to S^2_\kappa$ and structure group $SO(2)\otimes\mathbb{R}$, generated by $\{J_{12}, B\}$. Each irreducible 1D representation, $D_{\lambda,\beta}$, of the isotropy subgroup, $SO(2)\otimes\mathbb{R}$, of x_0 allows us to build up an associated vector bundle, $E_{D_{\beta,\lambda}}(S^2_\kappa, \pi_E, \mathbb{C})$, whose fibre is the support space, $\mathbb{C}$, of $D_{\beta,\lambda}$, where $\overline{SO}_\kappa(3)$ acts in a natural way. This action on the vector bundle translated to the linear space of bundle sections, i.e. Borel maps $f : S^2_\kappa \to E$ (which may be identified with the wavefunctions presented in section 3) defines the induced local representations, and the restriction to $SO_\kappa(3)$, by means of the section $s : SO_\kappa(3) \to \overline{SO}_\kappa(3)$, gives the local realizations. The gauge equivalence defined in the wavefunction space has its counterpart in terms of automorphisms of the vector bundle [32, 33].

It can be shown [33,34] that there is an invariant connection, Θ, under the action of $\overline{SO}_\kappa(3)$ on the principal bundle $\overline{SO}_\kappa(3)(S^2_\kappa, \pi, SO(2)\otimes\mathbb{R})$. The pull-back of Θ on the base space S^2_κ is represented on the associated vector bundle by the one-form $A = A_\mu(x)\,\mathrm{d}x^\mu$. The invariant condition expressed in terms of A is [34],

$$\mathcal{L}_X A - \mathrm{i}\,\mathrm{d}W = 0 \qquad \forall X \in so_\kappa(3) \tag{C.1}$$

where $\mathcal{L}_X$ denotes the Lie derivative of the vector field $X(x)$. Making use of the fields (B.3) we arrive at the potential

$$A_r = 0 \qquad A_\theta = b/\kappa - \lambda\cos\sqrt{\kappa}r. \tag{C.2}$$

The limit $\kappa \to 0$ is not defined but, if in agreement with the considerations of appendix B, we take $\lambda = b/\kappa \equiv \beta/\kappa$ we obtain a well behaved potential (5.2).

A significant property of an invariant connection on a principal bundle is the following one. Let us write the Lie commutators of the above-mentioned basis (2.5) of the Lie algebra $so_\kappa(3)$ in the form

$$[X_i, X_j] = c^k_{ij} X_k \tag{C.3}$$

where the structure constants c^k_{ij} are given in (2.1). Let $\overline{X}_i$ be the implementation of the vector fields X_i of $so_\kappa(3)$ as elements of $\overline{so}_\kappa(3)$ according to the vector field realization (3.2). Then, if we define the new set of generators $X^*_i = X_i(x)^\mu D_\mu$ with D_μ the covariant derivatives, i.e. the horizontal lifts of the fields $X_i = X^\mu_i(x)\partial_\mu$ of $so_\kappa(3)$, the following commutators are satisfied:

$$[\overline{X}_i, X^*_j] = c^k_{ij} X^*_k \tag{C.4}$$

where the coefficients c^k_{ij} coincide with the structure constants of (C.3).

The commutation relations (C.4) suggest that if $C_\kappa(X_j)$ denotes the Casimir of $SO_\kappa(3)$, then $C_\kappa(X^*_i)$ is a quadratic Casimir of $\overline{SO}_\kappa(3)$. So, it may differ with $\overline{C}_\kappa(\overline{X}_j)$ in a constant, but incidentally, in our case both coincide.

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